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ASYMPTOTIC STABILITY OF WEBSTER-LOKSHIN EQUATION

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Abstract. The Webster-Lokshin equation is a partial differential equation considered in this paper. It models the sound velocity in an acoustic domain. The dynamics contains linear fractional derivatives which can admit an infinite dimensional representation of diffusive type. The boundary conditions are described by impedance condition, which can be represented by two finite dimensional systems. Under the physical assumptions, there is a natural energy inequality. However, due to a lack of the precompactness of the solutions, the LaSalle invariance principle can not be applied. The asymptotic stability of the system is proved by studying the resolvent equation, and by using the Arendt-Batty stability condition.

1. **Introduction.** Our goal is to study the internal asymptotic stability of an infinite-dimensional linear model, namely a wave equation in a 1-D bounded domain. A classical undamped wave equation is known to be a conservative system, which can be described by a group of operators. On our more realistic model, there are two physical causes of dissipation: the damping at the boundaries and the internal damping.

First note that usual boundary conditions at the two ends of the pipe, either Dirichlet or Neumann boundary conditions are reflecting and account for a conservation of the wave energy; on the contrary, boundary conditions of *impedance* type are absorbing, and translate into dissipation of the wave energy, localized at the boundaries only. Most models of impedance are formulated in the frequency domain, and not the time domain; hence, since the impedances at stake, seen as transfer functions, happen to be positive real, one can apply the celebrated Kalman-Yakubovich-Popov lemma to build a realization, at least in finite dimension, see e.g. [11, 30] among other references. The latter realization happens to be of major help in deriving an energy balance, which will prove useful in the stability analysis of the coupled system.

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Second, there are different types of internal damping models for waves, corresponding to losses during the propagation; the most common ones are *fluid* or *viscous* damping, and *Kelvin-Voigt* or *Rayleigh* damping. Both these models are local in time, and allow for a straightforward semigroup formulation. Fluid damping corresponds to a uniform shift of the poles in the spectral domain, or to an exponential window in the time domain: the stability analysis of the system is quite elementary, see e.g. [19, Theorem 5.38]. With *Kelvin-Voigt* damping, the high-frequency modes are more heavily damped than the low-frequency ones, a situation which does occur in applications, making this model more realistic; the stability analysis can be performed by various methods, see e.g. [19, Section 4.3].

We are concerned here with a more complex damping model, known as damping of fractional order in time: causal fractional integrals or derivatives are non-local operators in time, which require an infinite-dimensional diagonal realization of diffusive type (see e.g. [31] and [27] independently, see also [25]) in order to get a semigroup formulation. An energy inequality is associated to this formulation, and a natural way to proceed to analyze stability would then be to use LaSalle's invariance principle; in infinite dimension though, the use of this principle requires to check the precompactness of the trajectories in the extended energy space. Unfortunately there is no simple way to check this property *a priori*, since the diffusive realization is made on an unbounded domain. This is the reason why we resort to some more sophisticated stability theorem, which requires the analysis of the spectrum of the generator of the extended semigroup, see [2] and [20].

The Lokshin equation has been introduced in [17, 18], and referred to in [13]. It has been established in [7] in the frequency domain and formulated in the time domain with fractional derivatives in [29]. For refined models of axi-symmetric pipes with varying cross-section, one can refer to [16] to understand what the best choice of variable z is for the Webster wave equation.

For the Lokshin model, there is a natural decay of the wave energy, as observed in [25], based on a spectral analysis first carried out in [24]. For the proof of stability, applying LaSalle's invariance principle requires a precompactness property (see e.g. [9, 8, 1]), which is not easy to get *a priori*.

Use of LaSalle invariance principle is possible when $\varepsilon = \eta = 0$, i.e. when there are no internal damping terms in the wave equation (1). More precisely, when $\varepsilon = \eta = 0$, the infinitesimal generator of the semigroup realizing the PDE under consideration in this paper has a compact resolvent. Therefore, in that case, LaSalle invariance principle can be applied, as in [1], [22, §.2], to prove asymptotic stability.

However, in presence of a diffusive realization either of standard or extended type, lack of precompactness is to be found, and we have to resort on [2, Stability theorem] and spectrum techniques (in our framework, it is equivalent to [20]).

See [26] for the stability proof of an ordinary differential equation coupled with a infinite dimension system coming from a diffusive representation. The paper [26] can be seen as the behavior of one mode of the wave equation coupled with an infinite dimensional system representing the fractional integrals and derivatives. The present paper can be seen as a generalization of [26] since the model under consideration is described by two coupled partial differential equations. This work has also a strong connection with the study of well-posedness of the Webster-Lokshin equation, as carried out in [14] and [15]. Some of the stability results proved here have been announced in [22].

The paper is organized as follows: in Section 2, the acoustic model is presented, together with the physical motivations; especially the notion of impedance is detailed. Then, in Section 3 realizations in the sense of systems theory are recalled: first, dissipative realizations for positive-real impedances are presented in § 3.1, and second dissipative realizations for positive pseudo-differential time-operators of diffusive type (such as fractional integral and derivatives) are given in § 3.2. Both these realizations enable to give an abstract formulation of the wave equation as a first order system in § 3.3. Its well-posedness is finally analyzed in § 3.4.

The core of the paper is Section 4, devoted to the study of asymptotic stability of the above model, once formulated as a first-order system. Since no compactness property can be found *a priori*, thus forbidding the use of LaSalle's invariance principle, then a refined analysis of the spectrum of the generator of the semigroup is carried out. The main result is Theorem 2, the proof of which heavily relies on Proposition 1, which is technical and will be proved in five steps.

Finally, Section 5 is devoted to conclusions on the problems treated in this paper and future works, including some possible generalizations and interesting open questions.

Notation: Given $z \in \mathbb{C}$, we denote its real part and its imaginary part by $\Re(z)$ and $\Im(z)$ respectively, and its complex conjugate by $\overline{z}$. Symbol $\mathbb{R}^+$ denotes the set of positive values. Finally x' or M' denotes the transpose of a vector or a matrix respectively, either real or complex.

2. **Acoustic model and physical motivations.** Consider an axi-symmetric duct between $z = 0$ and $z = 1$ with cross section radius $r(z)$ (satisfying $0 < r_0 \leq r \leq R_0 < \infty$ a. e.), where $r : [0, 1] \to \mathbb{R}$ is a function which is both bounded from below and essentially bounded (it does not need to be continuous), then the velocity potential ϕ (with appropriate scaling) satisfies the following equation:

$$\partial_t^2 \phi + (\eta(z)\, \partial_t^{\alpha} + \varepsilon(z)\, \partial_t^{-\beta})\, \partial_t \phi - \frac{1}{r^2(z)} \partial_z (r^2(z)\, \partial_z \phi) = 0, \tag{1}$$

for some $\alpha, \beta \in (0, 1)$ and $\varepsilon, \eta \in L^\infty(0, 1; \mathbb{R}^+)$. The terms in ∂_t^{α} and $\partial_t^{-\beta}$ model the effect of viscous and thermal losses at the lateral walls. The symbol $\partial_t^{-\beta}$ stands for the Riemann-Liouville fractional integral of order β, whereas ∂_t^{α} stands for the Riemann-Liouville fractional derivative of order α: both these operators are causal convolution products[1] with slowly decaying kernels, and their precise meaning will be given in Section 3.2 below when writing a realization.

We can reformulate (1) as a first order system in the (p, v) variables, where $p = \partial_t \phi$ is the pressure, and $v = -r^2\, \partial_z \phi$ is the volume velocity:

$$\partial_t p = -\frac{1}{r^2} \partial_z v - \varepsilon\, \partial_t^{-\beta} p - \eta\, \partial_t^{\alpha} p\ , \tag{2}$$

$$\partial_t v = -r^2\, \partial_z p\ , \tag{3}$$

To take into account the interaction with the exterior domain, one can add dynamical boundary conditions at $z = 0, 1$ that are of impedance type:

$$p_i(t) = \ell_i (h_i \star v_i)(t) \tag{4}$$

[1]Let us recall that the causal convolution product of two locally integrable functions h and v is defined by $(h \star v)(t) = \int_0^t h(t - \tau)\, v(\tau)\, \mathrm{d}\tau$ for almost every $t \geq 0$.

where $\star$ stands for the causal convolution product with respect to the time variable, $\ell_0 = -1$, and $\ell_1 = 1$, and where it is used the shorthand notation $p_i(t) = p(z = i, t)$ and $v_i(t) = v(z = i, t)$ for $i = 0, 1$. In (4), h_i are causal functions of time, and let us denote their Laplace transforms by $\mathcal{Z}_i(s)$ which are usually called *acoustic impedances*. Conditions (4) are formulated in the Laplace domain as

$$\hat{p}_i(s) = \ell_i \mathcal{Z}_i(s)\, \hat{v}_i(s) \quad \text{ for } i = 0, 1\ , \tag{5}$$

Such boundary conditions model very various absorbing boundary conditions. As an example, let us select $h_1 = Z_1 \delta_0$, where Z_1 is a given strictly positive and finite value in $\mathbb{R}$ and δ_0 is the Dirac function at $t = 0$: the limit case when $Z_1 = 0$, $\varepsilon = 0$, $\eta = 0$ would give, with (2) and (3), the Dirichlet boundary condition for the p-variable and the Neumann boundary condition for the v-variable; similarly, the other limit case when $Z_1 = \infty$, $\varepsilon = 0$, $\eta = 0$ would give the Neumann condition for the p-variable and the Dirichlet condition for the v-variable. Therefore, boundary condition (4) models time-varying Robin-type boundary conditions for the p and the v-variables: it does indeed interpolate between classical homogeneous Dirichlet and Neumann conditions (which are not addressed in this work), while preserving passivity.

The assumptions on the boundary conditions that will be needed in this work are collected in

Assumption 1. *The acoustic impedances $\mathcal{Z}_i$ in the boundary conditions (5), satisfy, for each $i = 0, 1$,*

1. *$\mathcal{Z}_i$ is a rational function without any pole at $s = 0$;*
2. *$\Re(\mathcal{Z}_i(s)) > 0$, $\forall s \in \mathbb{C}$, $\Re(s) \geq 0$;*

Under this assumption, item 1 will allow us to consider a realization of the boundary conditions in finite dimension. Item 2 means that $\mathcal{Z}_i(s)$ are *strictly positive real* in the sense of [19, ch. 5]. This latter item exludes both Dirichlet and Neumann boundary conditions; indeed, it imposes some dissipation of the energy, as will be seen from an energy balance law.

Assumption 1 holds as soon as h_i are real-valued functions of positive type, the algebraic structure of which are sums of Dirac measures and of causal polynomial-exponential functions (see e.g. [28]).

System (2)-(3)-(5) can be transformed into a first order system in time, using appropriate realizations for the pseudo-differential operators involved in this model:

- dissipative realizations for the boundary conditions written in terms of the positive-real impedances. This is done using Kalman-Yakubovich-Popov lemma in finite dimension, are recalled in § 3.1;
- dissipative realizations for the internal dynamics, more precisely for positive pseudo-differential time-operators of diffusive type, such as $\partial_t^{-\beta}$ and ∂_t^{α}, as presented in § 3.2.

This is the aim of next section together with the well-posedness of the model.

3. Realization and well-posedness.

3.1. Dissipative realizations for positive-real impedances.

Under item 1 of Assumption 1, there exists a *minimal* realization (A_i, B_i, C_i, d_i) with state x_i of *finite* dimension n_i ($A_i \in \mathbb{R}^{n_i \times n_i}$, $B_i \in \mathbb{R}^{n_i \times 1}$, $C_i \in \mathbb{R}^{1 \times n_i}$ and $d_i \in \mathbb{R}$), such that, for all $i = 0, 1$,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} x_i(t) &= A_i\, x_i + B_i\, v_i(t), \quad x_i(0) = 0 && (6) \\ \ell_i p_i(t) &= C_i\, x_i(t) + d_i\, v_i(t). && (7) \end{aligned}$$

Moreover, with item 2 of Assumption 1, using the Kalman-Yakubovich-Popov lemma (see e.g. [6, page 35] or [30]), there exists $P_i \in \mathbb{R}^{n_i \times n_i}$, $P_i = P_i' > 0$, such that the following energy balance holds, for each $T > 0$, and for any $v_i \in L^2([0, T); \mathbb{R})$,

$$\ell_i \int_0^T p_i(t)\, v_i(t)\, \mathrm{d}t = \frac{1}{2} x_i'(T)\, P_i\, x_i(T) + \frac{1}{2} \int_0^T \begin{pmatrix} x_i'(t) & v_i(t) \end{pmatrix} \mathcal{M}_i \begin{pmatrix} x_i(t) \\ v_i(t) \end{pmatrix} \mathrm{d}t, \qquad (8)$$

with $\mathcal{M}_i := \begin{pmatrix} -A_i' P_i - P_i A_i & C_i' - P_i B_i \\ C_i - B_i' P_i & 2 d_i \end{pmatrix} = \mathcal{M}_i' \geq 0$.

The right-hand side of (8) is split into two terms, a storage function evaluated at time T only, proportional to $x_i'(T) P_i x_i(T)$, and a dissipated energy on the time interval $(0, T)$, which involves the non-negative symmetric matrix $\mathcal{M}_i \in \mathbb{R}^{(n_i+1)\times(n_i+1)}$. We denote, for all $x = (x_0, x_1) \in \mathbb{R}^{n_0} \times \mathbb{R}^{n_1}$,

$$E_x := \frac{1}{2} x_0' P_0 \overline{x_0} + \frac{1}{2} x_1' P_1 \overline{x_1}.$$

Thus, when $\varepsilon = \eta = 0$, the global system (2)–(3)–(5) can be put in the abstract form $\frac{\mathrm{d}}{\mathrm{d}t} X = A\, X$, where:

$$A \begin{pmatrix} x_0 \\ x_1 \\ p \\ v \end{pmatrix} = \begin{pmatrix} A_0 x_0 + B_0\, v(z=0) \\ A_1 x_1 + B_1\, v(z=1) \\ -r^{-2} \partial_z v \\ -r^2 \partial_z p \end{pmatrix}; \qquad (9)$$

together with the boundary conditions $p(z=0) = -C_0 x_0 - d_0\, v(z=0)$ and $p(z=1) = C_1 x_1 + d_1\, v(z=1)$. In the sequel, we shall analyze the well-posedness of this system. Let us introduce the following spaces of complex-valued functions:

$$\begin{aligned} L_p^2 &:= \left\{ p, \int_0^1 |p|^2\, r^2(z)\, \mathrm{d}z < +\infty \right\}, \\ L_v^2 &:= \left\{ v, \int_0^1 |v|^2\, r^{-2}(z)\, \mathrm{d}z < +\infty \right\}, \\ H_p^1 &:= \left\{ p \in L_p^2, \int_0^1 (|p|^2 + |\partial_z p|^2)\, r^2\, \mathrm{d}z < +\infty \right\}, \\ H_v^1 &:= \left\{ v \in L_v^2, \int_0^1 (|v|^2 + |\partial_z v|^2)\, r^{-2}\, \mathrm{d}z < +\infty \right\}, \\ H &:= \mathbb{C}^{n_0} \times \mathbb{C}^{n_1} \times L_p^2 \times L_v^2, \end{aligned}$$

and

$$V := \mathbb{C}^{n_0} \times \mathbb{C}^{n_1} \times H_p^1 \times H_v^1.$$

For $i = 0, 1$, we equip $\mathbb{C}^{n_i}$ with the norm $x \mapsto x' P_i \overline{x}$, and we consider the L^2-norm on L_p^2 (resp. on L_v^2) with the weight r^2 (resp. r^{-2}). The hermitian product on H is thus written as, for all $X = (x_0, x_1, p, v)$ and $Y = (y_0, y_1, q, w)$ in H

$$(X, Y)_H = \sum_{i=0,1} x_i' P_i \overline{y_i} + \int_0^1 p(z) \overline{q(z)}\, r^2\, \mathrm{d}z + \int_0^1 v(z) \overline{w}(z)\, r^{-2}\, \mathrm{d}z.$$

The norm on H is denoted $\| \, \|_H$, i.e. for all $X = (x_0', x_1', p, v)' \in H$, $\|X\|_H^2 = \sum_{i=0,1} x_i' P_i \overline{x_i} + \int_0^1 |p|^2 \, r^2 \, \mathrm{d}z + \int_0^1 |v|^2 \, r^{-2} \, \mathrm{d}z$.

The domain of operator A reads:

$$D(A) = \left\{ (x_0', x_1', p, v)' \in V, \;\middle|\; \begin{array}{l} p(z=0) = -C_0 x_0 - d_0 \, v(z=0) \\ p(z=1) = C_1 x_1 + d_1 \, v(z=1) \end{array} \right\} .$$

Note that $D(A)$ is densely embedded in H.

Lemma 1. *Operator $-A$ is* monotone. *More precisely, the following equality holds, for all $X = (x_0', x_1', p, v)' \in D(A)$:*

$$\Re(-AX, \, X)_H = \frac{1}{2}\begin{pmatrix} x_0' & v(0) \end{pmatrix} \mathcal{M}_0 \overline{\begin{pmatrix} x_0 \\ v(0) \end{pmatrix}} + \frac{1}{2}\begin{pmatrix} x_1' & v(1) \end{pmatrix} \mathcal{M}_1 \overline{\begin{pmatrix} x_1 \\ v(1) \end{pmatrix}} . \tag{10}$$

Proof. Let us recall that, for all $(X, Y) \in H \times H$,

$$\Re(X, \, Y)_H = \frac{1}{4}\|X + Y\|_H^2 - \frac{1}{4}\|X - Y\|_H^2 \, .$$

Thus, for all $(X, Y) = ((x_0', x_1', p, v)', (y_0', y_1', q, w)') \in H \times H$, we have

$$\Re(X, \, Y)_H = \sum_{i=0,1} \Re(x_i' P_i \overline{y_i}) + \Re(\int_0^1 p\overline{q} \, r^2 \, \mathrm{d}z) + \Re(\int_0^1 v\overline{w} \, r^{-2} \, \mathrm{d}z) \, .$$

We compute, for all $X = (x_0', x_1', p, v)'$ in $D(A)$,

$$\begin{array}{rcl} \Re(AX, \, X)_H & = & \sum_{i=0,1} \Re(x_i' P_i \overline{(A_i x_i + B_i v(z=i))}) \\ & & -\Re(\int_0^1 p\overline{\partial_z v} \, \mathrm{d}z) - \Re(\int_0^1 v\overline{\partial_z p} \, \mathrm{d}z) \\ & = & \sum_{i=0,1} \Re(x_i' P_i \overline{(A_i x_i + B_i v(z=i))}) - \Re(\int_0^1 \partial_z(p\overline{v}) \, \mathrm{d}z) \\ & = & \sum_{i=0,1} \Re(x_i' P_i \overline{(A_i x_i + B_i v(z=i))}) \\ & & -\Re(p(z=1)\overline{v(z=1)}) + \Re(p(z=0)\overline{v(z=0)}) \\ & = & \sum_{i=0,1} \Re(x_i' P_i \overline{(A_i x_i + B_i v(z=i))}) - C_1 x_1 \overline{v(z=1)} - d_1 |v(z=1)|^2 \\ & & -C_0 x_0 \overline{v(z=0)} - d_0 |v(0)|^2 \end{array}$$

using the definition of $D(A)$.

Then, recalling that $\mathcal{M}_i := \begin{pmatrix} -A_i' P_i - P_i A_i & C_i' - P_i B_i \\ C_i - B_i' P_i & 2d_i \end{pmatrix}$, we can deduce that equation (10) holds. Now, since $\mathcal{M}_i \geq 0$ for $i = 0, 1$, we get that for all X in $D(A)$, $(-AX, \, X)_H \geq 0$, which prove that $-A$ is a monotone operator. $\square$

3.2. **Dissipative realizations for positive pseudo-differential time-operators of diffusive type.** For all $\gamma \in (0, 1)$, let us introduce the following kernel function $h_\gamma : \mathbb{R}^+ \to \mathbb{R}^+$, for all $t > 0$,

$$h_\gamma(t) := \int_0^\infty e^{-\xi t} \, \mathrm{d}M_\gamma(\xi) \tag{11}$$

where $\mathrm{d}M_\gamma(\xi) = \mu_\gamma(\xi) \, \mathrm{d}\xi$ with density $\mu_\gamma(\xi) := \frac{\sin(\gamma\pi)}{\pi} \, \xi^{-\gamma}$. Following e.g. [23], we can compute $h_\gamma(t) := \frac{1}{\Gamma(\gamma)} \, t^{\gamma-1}$, for $t > 0$, where Γ is the Euler gamma function. The definition of the Riemann-Liouville fractional integral of order γ of a locally integrable function v reads $I^\gamma v := h_\gamma \star v$, a causal convolution; it enjoys the nice property $I^{\gamma_1} \circ I^{\gamma_2} = I^{\gamma_1 + \gamma_2}$, $\forall \gamma_1, \gamma_2 > 0$. Since $I^1 v(t) := \int_0^t v(\tau) \, \mathrm{d}\tau$ is the integral of function v, it is convenient to denote it also by $\partial_t^{-1} v$; whence for any $\gamma \in (0, 1)$ the notation $\partial_t^{-\gamma} v$ will be used preferably throughout the paper for the fractional integral I^γ.

The following functional spaces will be of interest in the sequel,

$$\begin{array}{rcl} H_\gamma &=& L^2(\mathbb{R}^+;\mathbb{C};\mathrm{d}M_\gamma)\ , \\ V_\gamma &=& L^2\left(\mathbb{R}^+;\mathbb{C};(1+\xi)\,\mathrm{d}M_\gamma\right)\ , \\ \widetilde{H}_\gamma &=& L^2(\mathbb{R}^+;\mathbb{C};\xi\,\mathrm{d}M_\gamma)\ . \end{array}$$

We also introduce the notations $c_\gamma := \int_0^\infty \frac{\mathrm{d}M_\gamma(\xi)}{1+\xi} < \infty$. The condition $c_\gamma < \infty$ will be useful for the well-posedness condition.

3.2.1. *Standard diffusive representations for fractional integrals.* Let us define $\theta : \mathbb{R}^+ \to \mathbb{C}$ by, for all $t > 0$, $\theta(t) = h_\beta \star p\ (t)$, where $\star$ is the causal convolution product (the definition of which has been recalled after (4)). Note that since $h_\beta \in L^1(0,T)$ and $p \in L^2(0,T)$, we have $\theta \in L^2(0,T)$. Let us consider the realization of the input-output system $p \in L^2(0,T) \mapsto \theta \in L^2(0,T)$ (which will help us in yielding a representation of the fractional integral operator $\partial_t^{-\beta}$ introduced in (1), as done in Section 3.3 below, see e.g., [21, 27, 26])

$$\begin{array}{rcl} \partial_t\varphi(\xi,t) &=& -\xi\,\varphi(\xi,t)\ +\ p(t)\ , \qquad (12) \\ \varphi(\xi,0) &=& 0\ ,\ \forall\,\xi\in\mathbb{R}^+\ , \qquad (13) \\ \theta(t) &=& \displaystyle\int_0^{+\infty}\varphi(\xi,t)\,\mathrm{d}M_\beta(\xi)\ . \qquad (14) \end{array}$$

Using $h_\beta \star p \in L^2(0,T)$, we have $|h_\beta \star p(t)| < \infty$ for a.e. $t \in (0,T)$. Then using integral representation (11) of h_β and Fubini theorem, we get

$$\int_0^\infty\int_0^t e^{-\xi(t-\tau)}p(\tau)\,\mathrm{d}\tau\,\mathrm{d}M_\beta(\xi) = \int_0^t\int_0^\infty e^{-\xi(t-\tau)}\,\mathrm{d}M_\beta(\xi)p(\tau)\,\mathrm{d}\tau\ ,$$

and thus $\theta(t) = \partial_t^{-\beta}p(t)$ for a.e. $t \in (0,T)$. The following energy balance can be formally obtained:

$$\Re\Big(\int_0^T p(t)\,\overline{\theta}(t)\,\mathrm{d}t\Big) = \frac{1}{2}\int_0^\infty|\varphi(\xi,T)|^2\,\mathrm{d}M_\beta(\xi) + \int_0^T\int_0^\infty \xi\,|\varphi(\xi,t)|^2\,\mathrm{d}M_\beta(\xi)\,\mathrm{d}t\ . \qquad (15)$$

Similarly to (8), the right-hand side of (15) is split into two terms, a storage function evaluated at time T only, the following energy

$$E_\varphi(T) := \frac{1}{2}\|\varphi(T)\|^2_{H_\beta} = \frac{1}{2}\int_0^\infty|\varphi(\xi,T)|^2\,\mathrm{d}M_\beta(\xi),$$

and a dissipated energy on the time interval $(0,T)$.

3.2.2. *Extended diffusive representations for fractional derivatives.* For $\alpha \in (0,1)$, let us define $\tilde{\theta} : \mathbb{R}^+ \to \mathbb{C}$ by, for all $t > 0$, $\tilde{\theta}(t) = \frac{\mathrm{d}}{\mathrm{d}t}(h_{1-\alpha}\star p)\ (t)$. It can be shown that $\tilde{\theta}(t) = \partial_t^\alpha p$, the Riemann-Liouville fractional derivative of order α of function p; indeed the above formula reads $\partial_t^\alpha p(t) = \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t h_{1-\alpha}(t-\tau)\,p(\tau)\,\mathrm{d}\tau$, see e.g. [23]. Here, under regularity assumptions on function p, the property $\partial_t^{\alpha_1}\circ\partial_t^{\alpha_2} = \partial_t^{\alpha_1+\alpha_2}$ holds $\forall\alpha_1,\ \alpha_2 > 0$ such that $\alpha_1 + \alpha_2 < 1$.

Consider now the dynamical system with input $p \in H^1(0,T)$ and output $\widetilde{\theta} \in L^2(0,T)$:

$$
\begin{aligned}
\partial_t \widetilde{\varphi}(\xi,t) &= -\xi\, \widetilde{\varphi}(\xi,t) \,+\, p(t)\ , && (16)\\
\widetilde{\varphi}(\xi,0) &= 0\ ,\ \forall\, \xi \in \mathbb{R}^+\ , && (17)\\
\widetilde{\theta}(t) &= \int_0^\infty \partial_t \widetilde{\varphi}(\xi,t)\, \mathrm{d}M_{1-\alpha}(\xi) = \int_0^\infty \left[p(t) \,-\, \xi\, \widetilde{\varphi}(\xi,t)\right] \mathrm{d}M_{1-\alpha}(\xi)\ . && (18)
\end{aligned}
$$

Then it can be checked that $\widetilde{\theta}(t) = \partial_t^\alpha p(t)$ for a.e. $t \in (0,T)$. The following energy balance can be formally computed:

$$
\Re(\int_0^T p(t)\, \overline{\widetilde{\theta}}(t)\, \mathrm{d}t) = \frac{1}{2}\int_0^\infty |\widetilde{\varphi}(\xi,T)|^2 \xi\, \mathrm{d}M_{1-\alpha}(\xi) + \int_0^T \int_0^\infty |p - \xi\, \widetilde{\varphi}|^2\, \mathrm{d}M_{1-\alpha}\, \mathrm{d}t\ . \tag{19}
$$

Again the right-hand side of (19) is split into two terms, a storage function evaluated at time T only

$$
\widetilde{E}_{\widetilde{\varphi}}(T) := \frac{1}{2}\|\widetilde{\varphi}(T)\|^2_{\widetilde{H}_{1-\alpha}} = \frac{1}{2}\int_0^\infty |\widetilde{\varphi}(\xi,T)|^2 \xi\, \mathrm{d}M_{1-\alpha}(\xi),
$$

and a dissipated energy on the time interval $(0,T)$.

3.3. **An abstract formulation.** Now, using representations (6)-(7), (12)-(14), and (16)-(18), when $\varepsilon \neq 0$ and $\eta \neq 0$, the global system (2)-(3)-(5) can be realized into the first order differential equation in time

$$
\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{X} = \mathcal{A}\,\mathcal{X}, \tag{20}
$$

where $\mathcal{X} := (x_0', x_1', p, v, \varphi, \widetilde{\varphi})'$ and

$$
\mathcal{A}\begin{pmatrix} x_0 \\ x_1 \\ p \\ v \\ \varphi \\ \widetilde{\varphi} \end{pmatrix} := \begin{pmatrix} A_0 x_0 + B_0\, v(z=0) \\ A_1 x_1 + B_1\, v(z=1) \\ -r^{-2}\partial_z v - \varepsilon \int_0^{+\infty} \varphi\, \mathrm{d}M_\beta - \eta \int_0^{+\infty} [p \,-\, \xi\, \widetilde{\varphi}]\, \mathrm{d}M_{1-\alpha} \\ -r^2\, \partial_z p \\ -\xi\varphi + p \\ -\xi\widetilde{\varphi} + p \end{pmatrix}. \tag{21}
$$

The boundary conditions are, for each $i \in \{0,1\}$,

$$
p(i) = \ell_i(C_i x_i + d_i\, v(i)) \tag{22}
$$

must be taken into account in the functional spaces of the solutions. In the sequel, we shall analyze the well-posedness of this system. Let us compute, at least formally, the following energy balance

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\int_0^1 |p(z,t)|^2\; r^2(z)\,\mathrm{d}z + \frac{1}{2}\int_0^1 |v(z,t)|^2\; r^{-2}(z)\,\mathrm{d}z\right)\\
&+\frac{\mathrm{d}}{\mathrm{d}t}\left(E_x(t)\;+\;\int_0^1 \varepsilon(z)E_\varphi(z,t)\,r^2(z)\,\mathrm{d}z\;+\;\int_0^1 \eta(z)\widetilde{E}_{\widetilde{\varphi}}(z,t)\,r^2(z)\,\mathrm{d}z\right)\\
&\qquad\qquad = \;-\frac{1}{2}\begin{pmatrix}x_0' & v(0)\end{pmatrix}\mathcal{M}_0\overline{\begin{pmatrix}x_0\\ v(0)\end{pmatrix}} - \frac{1}{2}\begin{pmatrix}x_1' & v(1)\end{pmatrix}\mathcal{M}_1\overline{\begin{pmatrix}x_1\\ v(1)\end{pmatrix}}\\
&\qquad\qquad\quad\; -\int_0^1 \|\varphi\|^2_{\widetilde{H}_\beta}\,\varepsilon(z)\,r^2(z)\,\mathrm{d}z - \int_0^1 \|p-\xi\,\widetilde{\varphi}\|^2_{H_{1-\alpha}}\,\eta(z)\,r^2(z)\,\mathrm{d}z,
\end{aligned}
\tag{23}
$$

that will be proved in Theorem 1 below.

3.4. **Well-posedness of the global system.** We shall apply Lümer-Phillips theorem in order to show existence and uniqueness of solutions to (20).

According to identity (23), the natural *energy space* for the solution $\mathcal{X}$ would be the following Hilbert space:

$$
\mathcal{H} := \mathbb{C}^{n_0}\times\mathbb{C}^{n_1}\times L^2_p\times L^2_v\times L^2(0,1;H_\beta;\varepsilon\,r^2\,\mathrm{d}z)\times L^2(0,1;\widetilde{H}_{1-\alpha};\eta\,r^2\,\mathrm{d}z)\;,
$$

with norm, the square of which is equal to

$$
\begin{aligned}
\|\mathcal{X}\|^2_{\mathcal{H}} \;&=\; \sum_{i=0,1} x_i' P_i \overline{x_i} + \int_0^1 |p|^2\; r^2\,\mathrm{d}z + \int_0^1 |v|^2\, r^{-2}\,\mathrm{d}z\\
&\quad + \int_0^1 \big(\int_0^\infty |\varphi(\xi)|^2\,\mathrm{d}M_\beta(\xi)\big)\,\varepsilon(z)\,r^2(z)\,\mathrm{d}z\\
&\quad + \int_0^1 \big(\int_0^\infty \xi|\widetilde{\varphi}(\xi)|^2\,\mathrm{d}M_{1-\alpha}(\xi)\big)\,\eta(z)\,r^2(z)\,\mathrm{d}z\;.
\end{aligned}
$$

It is such that its hermitian product for $\mathcal{X}=(x_0',x_1',p,v,\varphi,\widetilde{\varphi})'$ and $\mathcal{Y}=(y_0',y_1',q,w,\psi,\widetilde{\psi})'$ satisfies:

$$
\begin{aligned}
\Re(\mathcal{X},\,\mathcal{Y})_{\mathcal{H}} \;&=\; \sum_{i=0,1}\Re(x_i'P_i\overline{y_i}) + \Re(p,\,q)_{L^2_p} + \Re(v,\,w)_{L^2_v}\\
&\quad + \int_0^1 \Re(\varphi,\,\psi)_{H_\beta}\,\varepsilon(z)\,r^2(z)\,\mathrm{d}z + \int_0^1 \Re(\widetilde{\varphi},\,\widetilde{\psi})_{\widetilde{H}_{1-\alpha}}\,\eta(z)\,r^2(z)\,\mathrm{d}z\;.
\end{aligned}
$$

We define the Hilbert space $\mathcal{V}$ as:

$$
\mathcal{V} := \mathbb{C}^{n_0}\times\mathbb{C}^{n_1}\times H^1_p\times H^1_v\times L^2(0,1;V_\beta;\varepsilon\,r^2\,\mathrm{d}z)\times L^2(0,1;\widetilde{H}_{1-\alpha};\eta\,r^2\,\mathrm{d}z)\;,
$$

where $L^2(0,1;V_\beta;\varepsilon\,r^2\,\mathrm{d}z)$ and $L^2(0,1;\widetilde{H}_{1-\alpha};\eta\,r^2\,\mathrm{d}z)$ are respectively the sets of functions φ and $\widetilde{\varphi}$ such that $\int_0^1 \|\varphi\|^2_{V_\beta}\,\varepsilon(z)\,r^2(z)\,\mathrm{d}z<\infty$ and $\int_0^1 \|\widetilde{\varphi}\|^2_{\widetilde{H}_{1-\alpha}}\,\eta(z)\,r^2(z)\,\mathrm{d}z<\infty$.

We set as domain of the operator $\mathcal{A}$, the space defined by:

$$
D(\mathcal{A}) := \left\{ (x_0',x_1',p,v,\varphi,\widetilde{\varphi})'\in\mathcal{V},\;\middle|\;
\begin{array}{l}
p(z=0) = -C_0x_0 - d_0\,v(z=0)\\
p(z=1) = C_1x_1 + d_1\,v(z=1)\\
(p-\xi\varphi)\in L^2(0,1;H_\beta;\varepsilon\,r^2\,\mathrm{d}z)\\
(p-\xi\widetilde{\varphi})\in L^2(0,1;V_{1-\alpha};\eta\,r^2\,\mathrm{d}z)
\end{array}
\right\}. \tag{24}
$$

Note that $D(\mathcal{A})$ is densely embedded in $\mathcal{H}$.

Lemma 2. *The operator $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ is well-defined.*

The proof of this lemma is postponed to Appendix A. The well-posedness of the global system is established in the following result.

Theorem 1. *Operator $\mathcal{A}$ generates a C^0-semigroup of contractions and, for each initial condition $\mathcal{X}_0 \in \mathcal{H}$, there exists a unique weak solution $\mathcal{X} \in C^0([0,+\infty);\mathcal{H}) \cap C^1([0,+\infty);D(\mathcal{A}^*)')$ to*

$$\left\{ \begin{array}{l} \dfrac{\mathrm{d}}{\mathrm{d}t}\mathcal{X}(t) = \mathcal{A}\,\mathcal{X}(t)\ ,\ \forall t > 0, \\ \mathcal{X}(0) = \mathcal{X}_0\ ; \end{array} \right. \tag{25}$$

where $D(\mathcal{A}^*)'$ is the topological dual of $D(\mathcal{A}^*)$ with respect to the pivot space $\mathcal{H}$.

Moreover, for each initial condition $\mathcal{X}_0 \in D(\mathcal{A})$, there exists a unique strong solution $\mathcal{X} \in C^0([0,+\infty);D(\mathcal{A})) \cap C^1([0,+\infty);\mathcal{H})$ to (25) and it satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{1}{2}\|\mathcal{X}(t)\|^2_{\mathcal{H}}\right\} = \Re(\mathcal{A}\mathcal{X}(t),\, \mathcal{X}(t))_{\mathcal{H}} \le 0\ . \tag{26}$$

Proof. We shall first prove the *monotonicity* of the operator $-\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$. Let $\mathcal{X} = (x_0', x_1', p, v, \varphi, \widetilde{\varphi})' \in D(\mathcal{A})$. Using (10), we have

$$\begin{aligned}
\Re(-\mathcal{A}X,\, X)_{\mathcal{H}} &= \frac{1}{2}\begin{pmatrix}x_0' & v(0)\end{pmatrix}\mathcal{M}_0\overline{\begin{pmatrix}x_0\\ v(0)\end{pmatrix}} + \frac{1}{2}\begin{pmatrix}x_1' & v(1)\end{pmatrix}\mathcal{M}_1\overline{\begin{pmatrix}x_1\\ v(1)\end{pmatrix}} \\
&\quad + \Re\int_0^1 \Big(\int_0^{+\infty} \varphi\,\mathrm{d}M_\beta\Big)\ \overline{p}\ \varepsilon\ r^2\,\mathrm{d}z \\
&\quad + \Re\int_0^1 \Big(\int_0^{+\infty} [p - \xi\tilde{\varphi}]\,\mathrm{d}M_{1-\alpha}\Big)\ \overline{p}\ \eta\ r^2\,\mathrm{d}z \\
&\quad + \int_0^1 \Re(\xi\varphi - p, \varphi)_{H_\beta}\ \varepsilon\ r^2\,\mathrm{d}z + \int_0^1 \Re(\xi\tilde{\varphi} - p, \tilde{\varphi})_{\tilde{H}_{1-\alpha}}\ \eta\ r^2\,\mathrm{d}z \\
&= \frac{1}{2}\begin{pmatrix}x_0' & v(0)\end{pmatrix}\mathcal{M}_0\overline{\begin{pmatrix}x_0\\ v(0)\end{pmatrix}} + \frac{1}{2}\begin{pmatrix}x_1' & v(1)\end{pmatrix}\mathcal{M}_1\overline{\begin{pmatrix}x_1\\ v(1)\end{pmatrix}} \\
&\quad + \int_0^1 \Big(\int_0^{+\infty} \xi\varphi\overline{\varphi}\,\mathrm{d}M_\beta\Big)\ \varepsilon\ r^2\,\mathrm{d}z \\
&\quad + \int_0^1 \Big(\int_0^{+\infty} [p - \xi\tilde{\varphi}]\ \overline{[p - \xi\tilde{\varphi}]}\,\mathrm{d}M_{1-\alpha}\Big)\ \eta\ r^2\,\mathrm{d}z \\
&= \frac{1}{2}\begin{pmatrix}x_0' & v(0)\end{pmatrix}\mathcal{M}_0\overline{\begin{pmatrix}x_0\\ v(0)\end{pmatrix}} + \frac{1}{2}\begin{pmatrix}x_1' & v(1)\end{pmatrix}\mathcal{M}_1\overline{\begin{pmatrix}x_1\\ v(1)\end{pmatrix}} \\
&\quad + \int_0^1 \|\varphi\|^2_{\tilde{H}_\beta}\,\varepsilon\, r^2\,\mathrm{d}z + \int_0^1 \|p - \xi\,\widetilde{\varphi}\|^2_{H_{1-\alpha}}\,\eta\, r^2\,\mathrm{d}z\ .
\end{aligned}$$

therefore $\Re(-\mathcal{A}\mathcal{X},\, \mathcal{X})_{\mathcal{H}} \ge 0$, for all $\mathcal{X}$ in $D(\mathcal{A})$ and the inequality in equation (26) will be fulfilled.

The maximality of $-\mathcal{A}$ has been already proved in [15] and [14, Theorem 2.2.1], by applying Lax-Milgram theorem: in this latter reference however, only strong solutions were examined (i.e. when X_0 lies in $D(\mathcal{A})$); moreover, only real-valued functional spaces were considered, whereas here, complex-valued functional spaces are allowed, and their study proves necessary for the spectral consideration of section 4 below.

An alternative and new proof provided here is to note that the maximality of $-\mathcal{A}$ is a special case of Proposition 1 below, with $\lambda = 1$.

From the monotonicity and maximality of the operator $-\mathcal{A}$, and applying Lümer-Phillips theorem (see e.g. [19, Theorem 2.27]), one concludes that $\mathcal{A}$ generates a C^0-semigroup of contraction, and that (26) holds. The existence and uniqueness of weak or strong solution can be found in e.g. [12, Chapter 3]. Concerning the regularity of the weak solutions, we refer to [32, Theorem 4.1.6], where the space X_{-1} is defined and identified as $D(\mathcal{A}^*)'$ in [32, Proposition 2.10.2]. □

4. **Asymptotic stability.** The aim of this section is to prove the following:

Theorem 2. *Under Assumption 1, we have the asymptotic stability property for (1) with the boundary conditions (5). It means that both the following properties hold:*

- *(Stability) for each initial condition $\mathcal{X}_0 \in \mathcal{H}$, the unique (weak) solution $\mathcal{X} \in C^0([0,+\infty);\mathcal{H})$ to (25) satisfies*
$$\|\mathcal{X}(t)\|_{\mathcal{H}} \leq \|\mathcal{X}_0\|_{\mathcal{H}} \ ;$$
- *(Attractivity) for each initial condition $\mathcal{X}_0 \in \mathcal{H}$, the unique (weak) solution $\mathcal{X} \in C^0([0,+\infty);\mathcal{H})$ to (25) satisfies*
$$\|\mathcal{X}(t)\|_{\mathcal{H}} \longrightarrow_{t\to\infty} 0 \ .$$

Let us first make a comment of physical nature on the above attractivity result: as soon as $\varepsilon > 0$ or $\eta > 0$, even with homogenous Dirichlet or Neumann boundary conditions instead of (5), this result would be true, as already proved in [22, §3.]. In the opposite case, when both $\varepsilon = 0$ and $\eta = 0$, Robin-type boundary conditions (5) with item 2 of Assumption 1 are required to prove the attractivity result, as already done in [22, §2.].

The proof of the stability part of Theorem 2 follows from the fact that $\mathcal{A}$ generates a semigroup of contractions, as claimed in Theorem 1.

Now, to prove the attractivity part of Theorem 2, let us apply [2, Stability theorem], that is recalled here:

Theorem 3. [2, Stability theorem] *Let us consider the infinitesimal generator $\mathcal{A}$ of a bounded $\mathcal{C}^0$-semigroup on a reflexive Banach space. Assume that no eigenvalue of $\mathcal{A}$ lies on the imaginary axis. If $\sigma(\mathcal{A}) \cap i\mathbb{R}$ is countable, then the semigroup generated by $\mathcal{A}$ is attractive, which means that the solutions $\mathcal{X}$ of the differential equation $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{X}(t) = \mathcal{A}\mathcal{X}(t)$ tend to 0 with $t \to \infty$.*

The attractivity part of Theorem 2 follows from this result, Theorem 1 and both the following lemmas:

Lemma 3. *We have*
$$\sigma(\mathcal{A}) \cap \{i\alpha,\ \alpha \in \mathbb{R},\ \alpha \neq 0\} = \emptyset \ .$$

Lemma 4. *$\lambda = 0$ is not an eigenvalue of $\mathcal{A}$.*

Let us first prove Lemma 4.

Proof. From (21) we get that $(x_0, x_1, p, q, \varphi, \tilde{\varphi}) \in \mathrm{Ker}(\mathcal{A}) \cap D(\mathcal{A})$ if and only if

$$\left\{\begin{array}{rcl} -A_0 x_0 - B_0 v(0) &=& 0 \\ -A_1 x_1 - B_1 v(1) &=& 0 \\ \frac{1}{r^2}\partial_z v + \varepsilon \int_0^\infty \varphi \,\mathrm{d}M_\beta(\xi) + \eta \int_0^\infty (p - \xi\tilde{\varphi})\,\mathrm{d}M_{1-\alpha}(\xi) &=& 0 \\ \partial_z p &=& 0 \\ \xi\varphi &=& p \\ \xi\tilde{\varphi} &=& p \end{array}\right. \tag{27}$$

First note that, in the special case $\varepsilon = \eta = 0$ when there is no internal damping, system (27) simplifies into four equations only, the two differential equations on v, p imply that they are constant functions, and the two algebraic equations on x_0, x_1 impose the boundary conditions on v, p; as in the general case below, these functions are found to be 0 thanks to item 2 of Assumption 1.
In the general case, the fourth equation of (27) implies that p is a constant function. Now the third and the fifth equation of (27) imply that $\frac{1}{r^2}\partial_z v + \varepsilon \int_0^\infty \frac{p}{\xi}\,\mathrm{d}M_\beta(\xi) = 0$. Since $\mathrm{d}M_\gamma(\xi) = \frac{\sin(\gamma\pi)}{\pi}\,\xi^{-\gamma}\,\mathrm{d}\xi$, the integral $\int_0^1 \frac{p}{\xi}\,\mathrm{d}M_\beta(\xi)$ converges if and only if $p = 0$. Thus p is identically equal to zero. This implies with the third equation of (27) that v is a constant function.

Let us recall the boundary conditions $p(z = 0) = -C_0 x_0 - d_0\, v(z = 0)$ and $p(z = 1) = C_1 x_1 + d_1\, v(z = 1)$. Since (6)-(7) is a minimal realization of $\mathcal{Z}_i$ and with item 1 of Assumption 1, $s = 0$ is not a pole of $\mathcal{Z}_i$ and thus A_i is invertible for each $i = 0, 1$. This gives $x_0 = -A_0^{-1} B_0 v(0)$ and thus $p(0) = (-C_0 A_0^{-1} B_0 - d_0))v(0) = -\mathcal{Z}_0(s = 0)v(0)$. Similarly, we have $p(1) = \mathcal{Z}_1(s = 0)v(1)$. Recall that, under item 2 of Assumption 1, the *acoustic impedances* $\mathcal{Z}_i(s)$ are *strictly positive real*, thus $p = v = 0$.

Now $\xi\varphi = 0$ and $\xi\tilde{\varphi} = 0$ are equivalent to $\varphi(\xi) = 0$ and $\tilde{\varphi}(\xi) = 0$, for all $\xi > 0$.

This concludes the proof of Lemma 4. □

Thus to complete the proof of Theorem 2, it remains to prove Lemma 3; it is a special case of the Proposition 1 below with $\lambda = i\omega \neq 0$.

Proposition 1. *For $\lambda \in \{i\omega,\ \omega \neq 0\} \cup \{\lambda \in \mathbb{R},\ \lambda > 0\}$, the resolvent operator $(\lambda I - \mathcal{A})^{-1}$ is a bounded operator from $\mathcal{H}$ to $\mathcal{H}$.*

Proof of Proposition 1. Let $\lambda \in \{i\omega,\ \omega \neq 0\} \cup \{\lambda \in \mathbb{R},\ \lambda > 0\}$. Let us first note that, for all $(x_0', x_1', p, v, \varphi, \tilde{\varphi})' \in D(\mathcal{A})$,

$$(\lambda I - \mathcal{A})\begin{pmatrix} x_0 \\ x_1 \\ p \\ v \\ \varphi \\ \tilde{\varphi} \end{pmatrix} = \begin{pmatrix} \lambda x_0 - A_0 x_0 - B_0 v(z = 0) \\ \lambda x_1 - A_1 x_1 - B_1 v(z = 1) \\ \lambda p + r^{-2}\partial_z v + \varepsilon \int_0^\infty \varphi dM_\beta + \eta \int_0^\infty (p - \xi\tilde{\varphi})\,\mathrm{d}M_{1-\alpha} \\ \lambda v + r^2 \partial_z p \\ \lambda\varphi + \xi\varphi - p \\ \lambda\tilde{\varphi} + \xi\tilde{\varphi} - p \end{pmatrix} \tag{28}$$

Let us consider the following resolvent equation, for all $(y_0', y_1', f, g, \chi, \tilde{\chi})' \in \mathcal{H}$, we look for some $(x_0', x_1', p, v, \varphi, \tilde{\varphi})' \in D(\mathcal{A})$ such that

$$\begin{pmatrix} y_0 \\ y_1 \\ f \\ g \\ \chi \\ \tilde{\chi} \end{pmatrix} = (\lambda I - \mathcal{A}) \begin{pmatrix} x_0 \\ x_1 \\ p \\ v \\ \varphi \\ \tilde{\varphi} \end{pmatrix} . \tag{29}$$

We divide the proof into five steps, as follows:

Step 1: solving (29) with respect to (x_0, x_1);
Step 2: solving (29) with respect to p;
Step 3: solving (29) with respect to v;
Step 4: solving (29) with respect to $(\varphi, \tilde{\varphi})$;
Step 5: checking that the solution $(x_0, x_1, p, v, \varphi, \tilde{\varphi})$ belongs to $D(\mathcal{A})$.

Step 1: solving (29) with respect to (x_0, x_1)**.** Recall that, under item 2 of Assumption 1, $\mathcal{Z}_i(s) = d_i + C_i(s\, I_{n_i} - A_i)^{-1}B_i$ is strictly positive real and the realization is minimal: thus, all eigenvalues of A_i are poles of $\mathcal{Z}_i(s)$, with strictly negative real parts. Thus $\lambda \notin \sigma(A_i)$ and one can solve the first two algebraic equations with respect to x_0, x_1,

$$x_i = (\lambda I_{n_i} - A_i)^{-1}(y_i + B_i v(i)) \tag{30}$$

for $i = 0, 1$. At this stage the function v is still to be determined.

Imposing from now on that we look for solutions belonging to $D(\mathcal{A})$, we have $p(z = i) = \ell_i(C_i x_i + d_i\, v(z = i))$ for each $i = 0, 1$, and thus, with (30)

$$p(i) = \ell_i\big(\mathcal{Z}_i(\lambda)v(i) + C_i(\lambda I_{n_i} - A_i)^{-1}y_i\big) \ , \tag{31}$$

and

$$v(i) = \frac{1}{\mathcal{Z}_i(\lambda)}\big(\ell_i p(i) - C_i(\lambda I_{n_i} - A)^{-1}y_i\big) \ . \tag{32}$$

Note that $\mathcal{Z}_i(\lambda) \neq 0$, since the acoustic impedances are *strictly* positive real, and $\lambda > 0$ or $\lambda = i\omega$ with $\omega \neq 0$.

Step 2: solving (29) with respect to p**.** With the last two equations of (29), we get

$$\varphi = \frac{\chi + p}{\lambda + \xi} \ , \tag{33}$$

and

$$\tilde{\varphi} = \frac{\tilde{\chi} + p}{\lambda + \xi} \ . \tag{34}$$

Both equations imply

$$\varphi = \frac{1}{\lambda + \xi} p + \frac{1}{\lambda + \xi} \chi \ , \tag{35}$$

and

$$p - \xi\tilde{\varphi} = \frac{\lambda}{\lambda + \xi} p - \frac{\xi}{\lambda + \xi} \tilde{\chi} \ . \tag{36}$$

Together with (28), the third equation of (29) yields

$$\big(\lambda + \varepsilon \int_0^\infty \frac{1}{\lambda+\xi}\,\mathrm{d}M_\beta(\xi) + \eta \int_0^\infty \frac{\lambda}{\lambda+\xi}\,\mathrm{d}M_{1-\alpha}(\xi)\big)p + r^{-2}\partial_z v = h \tag{37}$$

where h is defined by

$$h := f - \varepsilon \int_0^\infty \frac{\chi(\xi)}{\lambda+\xi}\,\mathrm{d}M_\beta(\xi) + \eta \int_0^\infty \frac{\xi\tilde{\chi}(\xi)}{\lambda+\xi}\,\mathrm{d}M_{1-\alpha}(\xi)\ .$$

Using Cauchy-Schwarz inequality (with $\chi \in H_\beta$ locally, and $\widetilde{\chi} \in \widetilde{H}_{1-\alpha}$ locally) and using the boundness of ε and η, one can easily check that $h \in L^2_p$.

We choose to solve (37) in the unknown variables (p, v) in a weak sense, in order to recover the regularity that is needed. To do that, we introduce $q \in H^1_p$. We have $r^2\partial_z q \in L^2_v$ and, with the fourth equation of (29), $g = \lambda v + r^2\partial_z p \in L^2_v$. Thus

$$-\lambda \int_0^1 v\overline{\partial_z q}\,\mathrm{d}z = -\int_0^1 g\overline{\partial_z q}\,\mathrm{d}z + \int_0^1 r^2\partial_z p\overline{\partial_z q}\,\mathrm{d}z\ .$$

By integrating in part in the first integral, we get

$$\lambda \int_0^1 \partial_z v\overline{q}\,\mathrm{d}z = -\int_0^1 g\overline{\partial_z q}\,\mathrm{d}z + \lambda(v(1)\overline{q(1)} - v(0)\overline{q(0)}) + \int_0^1 r^2\partial_z p\overline{\partial_z q}\,\mathrm{d}z\ . \tag{38}$$

Taking the hermitian product of (37) with $\lambda q r^2$, and using (38), we get

$$\begin{aligned}\lambda^2 \textstyle\int_0^1 p\overline{q}r^2\ \mathrm{d}z + \lambda \int_0^\infty \frac{1}{\lambda+\xi}\,\mathrm{d}M_\beta(\xi)\int_0^1 p\overline{q}\varepsilon r^2\ \mathrm{d}z + \lambda^2 \int_0^\infty \frac{1}{\lambda+\xi}\,\mathrm{d}M_{1-\alpha}(\xi)\int_0^1 \eta p\overline{q}r^2\ \mathrm{d}z\\ +\lambda(v(1)\overline{q(1)} - v(0)\overline{q(0)}) - \textstyle\int_0^1 g\overline{\partial_z q}\,\mathrm{d}z + \int_0^1 \partial_z p\overline{\partial_z q}r^2\ \mathrm{d}z = \lambda\int_0^1 h\overline{q}r^2\ \mathrm{d}z\ .\end{aligned} \tag{39}$$

With (32), we compute

$$\begin{aligned} v(1)\overline{q(1)} - v(0)\overline{q(0)} &= \frac{1}{\mathcal{Z}_1(\lambda)}p(1)\overline{q(1)} + \frac{1}{\mathcal{Z}_0(\lambda)}p(0)\overline{q(0)} \\ &\quad -\frac{1}{\mathcal{Z}_1(\lambda)}C_1(\lambda I_{n_1} - A_1)^{-1}y_1\overline{q(1)} \\ &\quad +\frac{1}{\mathcal{Z}_0(\lambda)}C_0(\lambda I_{n_0} - A_0)^{-1}y_0\overline{q(0)} \end{aligned} \tag{40}$$

Thus, using (40), we can rewrite (39) as an equation in the unknown $p \in H^1_p$, such that $a(p, q) = l(q)$ holds $\forall q \in H^1_p$, for some appropriate sesquilinear form a and anti-linear form l. In the case $\lambda > 0$, the complex version of Lax-Milgram applies, the coercivity of the underlying sesquilinear form a being garanteed thanks to $\Re(a(p, p)) \geq \min(1, \lambda^2)\ \|p\|_{H^1_p}$, the positivity of $\mathcal{Z}_i(\lambda)$ for $i = 0, 1$ when $\lambda > 0$, and the fact that $\lambda > 0$. This case is being used in Theorem 1, when proving the maximality of $\mathcal{A}$. But it is not general when $\lambda = i\omega$ with $\omega \neq 0$. The coercivity of a is certainly lost in that case. This is the reason why we resort to the Fredholm alternative, which is the proof that is presented here, which holds true in both cases.

With (40) at hand, we can rewrite (39) as

$$-(K^\lambda p, q)_{H^1_p} + (p, q)_{H^1_p} = l(q)$$

where

$$\begin{array}{rcl}(K^\lambda p, q)_{H^1_p} & := & (-\lambda^2+1)\int_0^1 p\overline{q}\; r^2\,\mathrm{d}z - \lambda\int_0^\infty \frac{1}{\lambda+\xi}\,\mathrm{d}M_\beta(\xi)\;\int_0^1 p\overline{q}\;\varepsilon\; r^2\;\;\mathrm{d}z \\ & & -\lambda^2\int_0^\infty \frac{1}{\lambda+\xi}\,\mathrm{d}M_{1-\alpha}(\xi)\;\int_0^1 p\overline{q}\;\eta\; r^2\;\;\mathrm{d}z \\ & & -\frac{\lambda}{\mathcal{Z}_1(\lambda)}\;p(1)\overline{q(1)} - \frac{\lambda}{\mathcal{Z}_0(\lambda)}\;p(0)\overline{q(0)}, \\ \\ l(q) & := & \lambda\int_0^1 h\,\overline{q}\,r^2\;\;\mathrm{d}z + \int_0^1 g\,\overline{\partial_z q}\;\;\mathrm{d}z + \mu_1\,\overline{q(1)} - \mu_0\,\overline{q(0)}\end{array}$$

with

$$\mu_i := \frac{\lambda}{\mathcal{Z}_i(\lambda)} C_i(\lambda I_{n_i} - A_i)^{-1} y_i\ .$$

Note that l is anti-linear in H^1_p. The continuity of l follows from trace theorem and the definition of the hermitian product in H^1_p. Thus, by Riesz representation theorem, there exists an $L\in H^1_p$ such that we have, for all $q\in H^1_p$,

$$l(q) = (L,q)_{H^1_p}.$$

Lemma 5. *Operator $K^\lambda : H^1_p \to H^1_p$ is compact in H^1_p.*

The proof of this lemma is postponed to Appendix B.

Now, with the Fredholm alternative, only two cases may occur:

- either 1 is an eigenvalue of K^λ:
- or 1 is not an eigenvalue of K^λ and then $(K^\lambda - I)^{-1}$ does exist and is continuous on H^1_p.

We may prove the following

Lemma 6. *The value* 1 *is not an eigenvalue of K^λ.*

Proof. To prove Lemma 6, we assume the converse and show a contradiction. If 1 is an eigenvalue, then there exists $p\in H^1_p$, $p\neq 0$, such that $(K^\lambda p,p)_{H^1_p} = (p,p)_{H^1_p}$. By definition of the operator K^λ, we get

$$\begin{array}{c}\lambda^2\int_0^1 |p|^2\; r^2\,\mathrm{d}z + \int_0^1 |\partial_z p|^2\; r^2\,\mathrm{d}z + \int_0^\infty \frac{\lambda(\overline{\lambda}+\xi)}{|\lambda+\xi|^2}\,\mathrm{d}M_\beta(\xi)\int_0^1 |p|^2\;\varepsilon\; r^2\,\mathrm{d}z \\ + \int_0^\infty \frac{\lambda^2(\overline{\lambda}+\xi)}{|\lambda+\xi|^2}\,\mathrm{d}M_{1-\alpha}(\xi)\int_0^1 |p|^2\eta r^2\;\;\mathrm{d}z + \frac{\lambda}{\mathcal{Z}_1(\lambda)}|p(1)|^2 + \frac{\lambda}{\mathcal{Z}_0(\lambda)}|p(0)|^2 = 0\ .\end{array} \tag{41}$$

Two cases may been inspected:

- if $\lambda > 0$, then, due to item 2 of Assumption 1, by inspecting the real part of the left-hand side of (41), a strictly positive value is obtained: this is a contradiction;
- else if $\lambda = i\omega$, with $\omega \neq 0$, then, by inspecting the imaginary part of (41) we get

$$\begin{array}{c}\Im(\lambda)\Big(\int_0^\infty \frac{\xi}{|\lambda+\xi|^2}\,\mathrm{d}M_\beta(\xi)\int_0^1 |p|^2\;\varepsilon\; r^2\,\mathrm{d}z \\ + \int_0^\infty \frac{|\lambda|^3}{|\lambda+\xi|^2}\,\mathrm{d}M_{1-\alpha}(\xi)\int_0^1 |p|^2\;\eta\; r^2\,\mathrm{d}z + \Re(\mathcal{Z}_1(\lambda))\frac{|p(1)|^2}{|\mathcal{Z}_1(\lambda)|^2} \\ + \Re(\mathcal{Z}_0(\lambda))\frac{|p(1)|^2}{|\mathcal{Z}_0(\lambda)|^2}\Big) = 0\end{array}$$

  which is a contradiction with $\Re(\mathcal{Z}_i(i\omega)) > 0$ (which follows from item 2 of Assumption 1), $\varepsilon(z)\geq 0$, and $\eta(z)\geq 0$, for all $z\in[0,1]$.

This concludes the proof of Lemma 6. $\square$

Combining the Fredholm alternative and Lemma 6, the map $\mathcal{H} \to H^1_p$, $(y_0, y_1, f, g, \chi, \tilde{\chi}) \mapsto p$ exists and is continuous.

Step 3: solving (29) with respect to v. With the fourth equation of (29) we can define $v = \frac{1}{\lambda}(g - r^2\,\partial_z p)$. It only belongs to L^2_v *a priori*. But with (37), $p \in H^1_p \subset L^2_p$ and $h \in L^2_p$, we have $\partial_z v \in L^2_p$ hence $v \in H^1_p$. Therefore the map $\mathcal{H} \to H^1_v$, $(y_0, y_1, f, g, \chi, \tilde{\chi}) \mapsto v$ is well-defined and continuous.

Step 4: solving (29) with respect to $(\varphi, \tilde{\varphi})$. In order to check that the unique solution X belongs to $\mathcal{V}$, we need to prove that $\varphi \in L^2(0,1; V_\beta; \varepsilon\, r^2\,\mathrm{d}z)$ and $\widetilde{\varphi} \in L^2(0,1; \widetilde{H}_{1-\alpha}; \eta\, r^2\,\mathrm{d}z)$, using $p \in H^1_p \subset L^2_p$, $\chi \in L^2(0,1; H_\beta; \varepsilon\, r^2\,\mathrm{d}z)$ and $\widetilde{\chi} \in L^2(0,1; \widetilde{H}_{1-\alpha}; \eta\, r^2\,\mathrm{d}z)$.

Note that, for all $\xi > 0$,

$$\frac{1}{|\lambda + \xi|} \leq \max\left(1, \frac{1}{|\lambda|}\right) \frac{\sqrt{2}}{1+\xi}\ . \tag{42}$$

and let $M_\lambda := \sqrt{2}\max\left(1, \frac{1}{|\lambda|}\right)$.

Recall (35). On the first hand, due to (42), since $\|\frac{1}{1+\xi}\|^2_{V_\beta} = c_\beta$ one has

$$\begin{array}{rcl} \|\frac{1}{\lambda+\xi}\, p\|^2_{L^2(0,1;V_\beta;\varepsilon\, r^2\,\mathrm{d}z)} & \leq & M_\lambda \|\frac{1}{1+\xi}\, p\|^2_{L^2(0,1;V_\beta;\varepsilon\, r^2\,\mathrm{d}z)} \\ & \leq & M_\lambda c_\beta\, \|\varepsilon\|_\infty\, \|p\|^2_{L^2_p}; \end{array}$$

on the other hand, since $\|\frac{1}{1+\xi}\,\chi\|^2_{V_\beta} = \|\frac{1}{\sqrt{1+\xi}}\,\chi\|^2_{H_\beta}$, then one has

$$\begin{array}{rcl} \|\frac{1}{\lambda+\xi}\,\chi\|^2_{L^2(0,1;V_\beta;\varepsilon\, r^2\,\mathrm{d}z)} & \leq & M_\lambda \|\frac{1}{1+\xi}\,\chi\|^2_{L^2(0,1;V_\beta;\varepsilon\, r^2\,\mathrm{d}z)} \\ & \leq & M_\lambda \|\chi\|^2_{L^2(0,1;H_\beta;\varepsilon\, r^2\,\mathrm{d}z)}. \end{array}$$

Similar considerations apply to $\widetilde{\varphi} = \frac{1}{\lambda+\xi}\, p + \frac{1}{\lambda+\xi}\,\widetilde{\chi}$; indeed $\|\frac{1}{1+\xi}\|^2_{\widetilde{H}_{1-\alpha}} \leq c_{1-\alpha}$ implies

$$\begin{array}{rcl} \|\frac{1}{\lambda+\xi}\, p\|^2_{L^2(0,1;\widetilde{H}_{1-\alpha};\eta\, r^2\,\mathrm{d}z)} & \leq & M_\lambda \|\frac{1}{1+\xi}\, p\|^2_{L^2(0,1;\widetilde{H}_{1-\alpha};\eta\, r^2\,\mathrm{d}z)} \\ & \leq & M_\lambda c_{1-\alpha}\, \|\eta\|_\infty\, \|p\|^2_{L^2_p}; \end{array}$$

whereas we have

$$\|\frac{1}{\lambda+\xi}\,\widetilde{\chi}\|^2_{L^2(0,1;\widetilde{H}_{1-\alpha};\eta\, r^2\,\mathrm{d}z)} \leq M_\lambda \|\widetilde{\chi}\|^2_{L^2(0,1;\widetilde{H}_{1-\alpha};\eta\, r^2\,\mathrm{d}z)}.$$

Thus, with (37), the map

$$\begin{array}{rcl} \mathcal{H} & \to & L^2(0,1; V_\beta; \varepsilon\, r^2\,\mathrm{d}z) \times L^2(0,1; \widetilde{H}_{1-\alpha}; \eta\, r^2\,\mathrm{d}z) \\ (y_0, y_1, f, g, \chi, \tilde{\chi}) & \mapsto & (\varphi, \tilde{\varphi}) \end{array}$$

is also well-defined and continuous.

Step 5: checking that the solution $(x_0, x_1, p, v, \varphi, \tilde{\varphi})$ belongs to $D(\mathcal{A})$. At the end of Step 4, it is proved that there exists a unique $(x_0, x_1, p, v, \varphi, \tilde{\varphi})$ in $\mathcal{V}$ solving (29). It remains to show that this solution belongs to $D(\mathcal{A})$.

It has already been taken explicitely into account that $p(z = 0) = -C_0 x_0 - d_0\, v(z = 0)$, $p(z = 1) = C_1 x_1 + d_1\, v(z = 1)$ in Step 1 of this proof.

What remains to be proved is that $p - \xi\varphi \in L^2(0,1; H_\beta; \varepsilon\, r^2\, dz)$ and $p - \xi\,\widetilde{\varphi} \in L^2(0,1; V_{1-\alpha}; \eta\, r^2\, dz)$. Let us check that successively.

• From $\xi\,\varphi - p = -\frac{\lambda}{\lambda+\xi}\,p + \frac{\xi}{\lambda+\xi}\,\chi$ and (42), one easily deduces that $(p - \xi\varphi) \in L^2(0,1;H_\beta;\varepsilon\,r^2\,dz)$, since

$$\|\frac{\lambda}{\lambda+\xi}\,p\|^2_{L^2(0,1;H_\beta;\varepsilon\,r^2\,dz)} \leq |\lambda| M_\lambda c_\beta\ \|\varepsilon\|_\infty\ \|p\|^2_{L^2_p},$$

and using $\frac{\xi}{1+\xi} \leq 1$,

$$\|\frac{\xi}{\lambda+\xi}\,\chi\|^2_{L^2(0,1;H_\beta;\varepsilon\,r^2\,dz)} \leq M_\lambda \|\chi\|^2_{L^2(0,1;H_\beta;\varepsilon\,r^2\,dz)}.$$

• One checks that $\xi\,\widetilde{\varphi} - p = -\frac{\lambda}{\|a+\xi}\,p + \frac{\xi}{\lambda+\xi}\,\widetilde{\chi} \in L^2(0,1;V_{1-\alpha};\eta\,r^2\,dz)$ by firstly noting that

$$\|\frac{\lambda}{\lambda+\xi}\,p\|^2_{L^2(0,1;V_{1-\alpha};\eta\,r^2\,dz)} \leq |\lambda| M_\lambda c_{1-\alpha}\ \|\eta\|_\infty\ \|p\|^2_{L^2_p},$$

and secondly using $\|\frac{\xi}{\lambda+\xi}\,\widetilde{\chi}\|^2_{V_{1-\alpha}} \leq M_\lambda \|\sqrt{\frac{\xi}{1+\xi}}\,\widetilde{\chi}\|^2_{\widetilde{V}_{1-\alpha}} \leq M_\lambda \|\widetilde{\chi}\|^2_{\widetilde{V}_{1-\alpha}}$ to deduce

$$\|\frac{\xi}{\lambda+\xi}\,\widetilde{\chi}\|^2_{L^2(0,1;V_{1-\alpha};\eta\,r^2\,dz)} \leq M_\lambda \|\widetilde{\chi}\|^2_{L^2(0,1;\widetilde{V}_{1-\alpha};\eta\,r^2\,dz)}.$$

This concludes the proof of Proposition 1.

5. **Conclusion.** In this paper, the stability of Webster-Lokshin equation has been proven, under physically relevant assumptions. This equation models the sound propagation in a bounded acoustic domain. A representation in an infinite dimensional space has been used to represent the fractional integrals and fractional derivatives, whereas the boundary conditions of the partial differential equation are given by a finite dimensional dynamics. Exploiting the energy decay is not sufficient to prove the stability since the LaSalle invariance principle did not apply. However a study of the resolvent equation is fruitful when using the Arendt-Batty stability condition.

This work leaves many questions open. In particular it could be interesting to study the speed of convergence as the time goes to the infinity. More precisely, even if it is known that exponential stability does not hold (see e.g. [26, Remark 2.7] for the one-dimensional case, or [21] for fractional differential equations), employing the resolvent equation approach and applying [4, 5], we might be able to characterize the speed of decay.

As a possible illustration of previous research line, designing a numerical scheme as in [14, Chapter 3] may me fruitful to check the convergence speed of the energy.

Another question is to relax item 1 of Assumption 1, and make use of a dissipative realization in an infinite-dimensional space, as in [3], for a passive non-rational impedance.

More difficult questions of theoretical nature then arise when dealing with nonlinear PDE models, such as the Burgers-Lokshin model, which is being used in musical acoustics to model brassy effects in wind instruments: both nonlinearity and fractional derivatives are to be found in this wave equation. But different techniques to study asymptotic stability, if any, will have to be used, following e.g. [10] and references therein. Again an energy balance can be fruitful to compute candidate Lyapunov functions.

Appendix A. **Proof of Lemma 2.** To prove that $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ is well-defined, let us consider each term of (21) separately.

- for the first two components of equation (21), due to the trace theorem
$$|v(z=i)| \le c_0 \, \|v\|_{H^1}$$
for some positive constant c_0, therefore $|v(z=i)| \le c_0 \, \|r^2\|_{L^\infty} \|v\|_{H^1_v}$;
- for the third component, on the one hand we have
$$\|r^{-2} \, \partial_z v\|_{L^2_p} = \sqrt{\int_0^1 (\partial_z v)^2 r^{-2} \, \mathrm{d}z} \le \|v\|_{H^1_v} \; .$$
One the other hand, using Schwarz inequality
$$\left| \int_0^\infty \varphi \, \mathrm{d}M_\beta \right|^2 \le \left(\int_0^\infty |\varphi| \, \mathrm{d}M_\beta \right)^2 \le c_\beta \, \int_0^\infty (1+\xi)|\varphi|^2 \, \mathrm{d}M_\beta \; ;$$
hence
$$\left\| \varepsilon \, \int_0^\infty \varphi \, \mathrm{d}M_\beta \right\|^2_{L^2_p} \le c_\beta \, \|\varepsilon\|_{L^\infty} \, \|\varphi\|^2_{L^2(0,1;V_\beta;\varepsilon r^2 \, \mathrm{d}z)}.$$
Finally, using again Schwarz inequality, we have
$$\begin{array}{rcl} \left| \int_0^\infty (p - \xi\tilde{\varphi}) \, \mathrm{d}M_{1-\alpha} \right|^2 & \le & \left| \int_0^\infty (p - \xi\tilde{\varphi}) \, \mathrm{d}M_{1-\alpha} \right|^2 \\ & \le & \left(\int_0^\infty (1+\xi)|p - \xi\tilde{\varphi}|^2 \, \mathrm{d}M_{1-\alpha}(\xi) \right) \left(\int_0^\infty \frac{1}{1+\xi} \, \mathrm{d}M_{1-\alpha}(\xi) \right) \; . \end{array}$$
Hence
$$\begin{array}{rcl} \left\| \eta \, \int_0^\infty (p - \xi\widetilde{\varphi}) \, \mathrm{d}M_{1-\alpha} \right\|^2_{L^2_p} & = & \int_0^1 \eta^2(z) | \int_0^\infty (p - \xi\widetilde{\varphi}) \, \mathrm{d}M_{1-\alpha}|^2 r^2(z) \, \mathrm{d}z \\ & \le & c_{1-\alpha} \, \|\eta\|_{L^\infty} \, \int_0^1 \|p - \xi\tilde{\varphi}\|^2_{V_{1-\alpha}} \eta(z) r^2(z) \, \mathrm{d}z \\ & \le & c_{1-\alpha} \, \|\eta\|_{L^\infty} \, \|p - \xi\widetilde{\varphi}\|^2_{L^2(0,1;V_{1-\alpha};\eta \, r^2 \, \mathrm{d}z)}; \end{array}$$
- for the fourth component, obviously, $\|r^2 \partial_z p\|_{L^2_v} \le \|p\|_{H^1_p}$;
- for the fifth component, $(p - \xi\varphi) \in L^2(0,1;H_\beta;\varepsilon \, r^2 \, \mathrm{d}z)$, by definition of $D(\mathcal{A})$;
- for the sixth component, since $V_{1-\alpha} \subset \widetilde{H}_{1-\alpha}$, we simply have
$$\|p - \xi\widetilde{\varphi}\|_{L^2(0,1;\widetilde{H}_{1-\alpha};\eta \, r^2 \, \mathrm{d}z)} \le \|p - \xi\widetilde{\varphi}\|_{L^2(0,1;V_{1-\alpha};\eta \, r^2 \, \mathrm{d}z)}.$$

Hence, there exists a $C > 0$, such that for all $X \in D(\mathcal{A})$, $\|\mathcal{A}X\|_{\mathcal{H}} \le C\|X\|_{\mathcal{V}}$. This concludes the proof of Lemma 2.

Appendix B. **Proof of Lemma 5.** We have $K^\lambda = K_0^\lambda + K_1^\lambda + K_2^\lambda$ where K_0^λ, K_1^λ and K_2^λ are three operators on H^1_p respectively defined by, for all $(p,q) \in H^1_p \times H^1_p$,
$$\begin{array}{rcl} (K_i^\lambda p, q)_{H^1_p} & = & -\frac{\lambda}{\mathcal{Z}_i(\lambda)} p(i) \overline{q(i)} \; , \; \forall i = 0,1 \\ (K_2^\lambda p, q)_{H^1_p} & = & \int_0^1 \Omega_\lambda(z) \; p \; \bar{q} \; r^2 \, \mathrm{d}z \end{array}$$
where $\Omega(z) := -\lambda^2 + 1 - \lambda \int_0^\infty \frac{1}{\lambda+\xi} \, \mathrm{d}M_\beta(\xi) \, \varepsilon(z) - \lambda^2 \int_0^\infty \frac{1}{\lambda+\xi} \, \mathrm{d}M_{1-\alpha}(\xi) \, \eta(z)$.

The continuity K_i^λ, for $i = 0,1$, is due to the continuity of the trace function. Moreover, by Riesz representation theorem, $(K_i^\lambda p, q)_{H^1_p} = (p, \varpi_i)_{H^1_p} \overline{(q, \varpi_i)_{H^1_p}}$ for a suitable $\varpi_i \in H^1_p$. Therefore $\{\varpi_i\}^\perp = \mathrm{Ker}(K_i^{\lambda\star})$ hence $\{\varpi_i\} = \overline{\mathrm{Im}(K_i^\lambda)}$. Thus for $i = 0,1$, K_i^λ is of rank one, hence compact.

Now we note that, for all $(p,q) \in H^1_p \times H^1_p$,
$$|(K_2^\lambda p, q)_{H^1_p}| \le C\|p\|_{H^1_p} \|q\|_{H^1_p}$$
for a suitable $C > 0$. Thus K_2^λ is continuous on H^1_p.

First, using Riesz representation theorem, it can easily be proved that K_2^λ is defined as a bounded operator on H_p^1.

Second, taking $q = K_2^\lambda p$ in the definition, and using Cauchy-Schwarz inequality, we prove the following:

$$\forall p \in H_p^1, \quad \|K_2^\lambda p\|_{H_p^1} \leq \|\Omega_\lambda\|_{L^\infty} \, \|p\|_{L_p^2} \, .$$

Now, let $(p_n)_{n\in\mathbb{N}}$ be a *bounded* sequence in H_p^1; since $(0,1)$ is a bounded domain, thanks to Rellich-Kondrakov theorem, we can extract a subsequence $(p_n)_{n'\in\mathbb{N}}$ that is convergent in L_p^2. Hence,

$$\|K_2^\lambda p_{n'} - K_2^\lambda p_{m'}\|_{H_p^1} \leq \|\Omega_\lambda\|_{L^\infty} \, \|p_{n'} - p_{m'}\|_{L_p^2} \, ,$$

meaning that $(K_2^\lambda p_{n'})_{n'\in\mathbb{N}}$ is a Cauchy sequence in H_p^1; since the Hilbert space H_p^1 is complete, it follows that the sequence $(K_2^\lambda p_{n'})_{n'\in\mathbb{N}}$ is convergent. Hence, operator $K_2^\lambda : H_p^1 \to H_p^1$ is *compact*.

Therefore the operator $K^\lambda : H_p^1 \to H_p^1$ is a compact operator in H_p^1 since it is the sum of three compact operators.

This concludes the proof of Lemma 5.

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